2022

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AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 1

- ☑ Scoring Guidelines
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Part A (AB or BC): Graphing calculator required

Question 1 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

Model Solution	Scoring

(a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

The total number of vehicles that arrive at the toll plaza from 6 A.M. to 10 A.M. is given by $\int_1^5 A(t)\,dt$.	Answer	**1 point**

Scoring notes:

- The response must be a definite integral with correct lower and upper limits to earn this point.
- Because $|A(t)| = A(t)$ for $1 \le t \le 5$, a response of $\int_1^5 \left|450\sqrt{\sin(0.62t)}\right| dt$ or $\int_1^5 |A(t)|\,dt$ earns the point.
- A response missing dt or using dx is eligible to earn the point.
- A response with a copy error in the expression for $A(t)$ will earn the point only in the presence of $\int_1^5 A(t)\,dt$.

Total for part (a) **1 point**

(b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

$\text{Average} = \frac{1}{5-1}\int_1^5 A(t)\,dt = 375.536966$ The average rate at which vehicles arrive at the toll plaza from 6 A.M. to 10 A.M. is 375.537 (or 375.536) vehicles per hour.	Uses average value formula: $\frac{1}{b-a}\int_a^b A(t)\,dt$	**1 point**
	Answer	**1 point**

Scoring notes:

- The use of the average value formula, indicating that $a = 1$ and $b = 5$, can be presented in single or multiple steps to earn the first point. For example, the following response earns both points: $\int_1^5 A(t)\,dt = 1502.147865$, so the average value is 375.536966.
- A response that presents a correct integral along with the correct average value, but provides incorrect or incomplete communication, earns 1 out of 2 points. For example, the following response earns 1 out of 2 points: $\int_1^5 A(t)\,dt = 1502.147865 = 375.536966$.
- The answer must be correct to three decimal places. For example, $\frac{1}{5-1}\int_1^5 A(t)\,dt = 375.536966 \approx 376$ earns only the first point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{4}\int_1^5 A(t)\,dt = 79.416068$.
- Special case: $\frac{1}{5}\int_1^5 A(t)\,dt = 300.429573$ earns 1 out of 2 points.

Total for part (b) 2 points

(c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

$A'(1) = 148.947272$	Considers $A'(1)$	**1 point**
Because $A'(1) > 0$, the rate at which the vehicles arrive at the toll plaza is increasing.	Answer with reason	**1 point**

Scoring notes:

- The response need not present the value of $A'(1)$. The second line of the model solution earns both points.
- An incorrect value assigned to $A'(1)$ earns the first point (but will not earn the second point).
- Without a reference to $t = 1$, the first point is earned by any of the following:
  - 148.947 accurate to the number of decimals presented, with zero up to three decimal places (i.e., 149, 148, 148.9, 148.95, or 148.94)
  - $A'(t) = 148.947$ by itself
- To be eligible for the second point, the first point must be earned.
- To earn the second point, there must be a reference to $t = 1$.
- Degree mode: $A'(1) = 23.404311$

Total for part (c) 2 points

(d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\, dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

$N'(t) = A(t) - 400 = 0$
$\Rightarrow A(t) = 400 \Rightarrow t = 1.469372,\ t = 3.597713$

$a = 1.469372$
$b = 3.597713$

Considers $N'(t) = 0$	**1 point**
$t = a$ and $t = b$	**1 point**

t	$N(t) = \int_a^t (A(x) - 400)\, dx$
a	0
b	71.254129
4	62.338346

The greatest number of vehicles in line is 71.

Answer	**1 point**
Justification	**1 point**

Scoring notes:

- It is not necessary to indicate that $A(t) = 400$ to earn the first point, although this statement alone would earn the first point.
- A response of "$A(t) \geq 400$ when $1.469372 \leq t \leq 3.597713$" will earn the first 2 points. A response of "$A(t) \geq 400$" along with the presence of exactly one of the two numbers above will earn the first point, but not the second. A response of "$A(t) \geq 400$" by itself will not earn either of the first 2 points.
- To earn the second point the values for a and b must be accurate to the number of decimals presented, with at least one and up to three decimal places. These may appear only in a candidates table, as limits of integration, or on a number line.
- A response with incorrect notation involving t or x is eligible to earn all 4 points.
- A response that does not earn the first point is still eligible for the remaining 3 points.
- To earn the third point, a response must present the greatest number of vehicles. This point is earned for answers of either 71 or 71.254*** only.
- A correct justification earns the fourth point, even if the third point is not earned because of a decimal presentation error.
- When using a Candidates Test, the response must include the values for $N(a)$, $N(b)$, and $N(4)$ to earn the fourth point. These values must be correct to the number of decimals presented, with up to three decimal places. (Correctly rounded integer values are acceptable.)
- Alternate solution for the third and fourth points:

  For $a \leq t \leq b$, $A(t) \geq 400$. For $b \leq t \leq 4$, $A(t) \leq 400$.

  Thus, $N(t) = \int_a^t (A(x) - 400)\, dx$ is greatest at $t = b$.

  $N(b) = 71.254129$, and the greatest number of vehicles in line is 71.
- Degree mode: The response is only eligible to earn the first point because in degree mode $A(t) < 400$.

Total for part (d) 4 points

Total for question 1 9 points

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Answer QUESTION 1 parts (a) and (b) on this page.

Response for question 1(a)

$\int_1^5 450\sqrt{\sin(0.62t)}\,dt$

Response for question 1(b)

$\frac{1}{5-1}\int_1^5 450\sqrt{\sin(0.62t)}\,dt = 375.537$ vehicles per hour

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Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

$A'(1) = 148.947$ vehicles per hour per hour

The rate at which vehicles arrive at the toll plaza at 6 A.M. $(t=1)$ is increasing because the rate of change of $A(t)$ at $t=1$ is positive.

Response for question 1(d)

$N'(t) = A(t) - 400 = 0$ when $t = 3.59771$ hours

$A(t) = 450\sqrt{\sin(0.62t)} = 400$, $t = 1.46937$

t	$N(t)$
1.46937	0
3.59771	71.2541
4	62.3383

The greatest number of vehicles in line is 71 vehicles at $t = 3.59771$ hours because $N(t)$ achieves a relative maximum at $t = 3.59771$, and since $t = 3.59771$ is the only critical number on the given interval, the relative maximum is the absolute maximum.

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Q5112/5

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Answer QUESTION 1 parts (a) and (b) on this page.

Response for question 1(a)

$\int_1^5 A(t)\,dt$

$A(t) = 450\sqrt{\sin(0.62t)}$

Response for question 1(b)

$\frac{1}{5-1}\int_1^5 A(t)\,dt$

$= 375.537 \ \frac{\text{vehicles}}{\text{hour}}$

the

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5106/4

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

$A(t)=450\sqrt{\sin(0.62t)}$

$A'(1)=148.947\ldots$

At 6 AM or $t=1$ (time), the rate at which vehicles arrive @ the toll plaza is increasing because $A'(1)>0$.

Response for question 1(d)

Abs. max of # vehicles.

$N(t)=\int_a^t (A(t)-400)\,dt$

$a\le t\le 4$

$\sqrt{\sin(0.62t)}=1$

$t=\boxed{1.612}$

$N'(t)=A(t)-400$

$N'(t)=0$ or DNE

a | B | 4

$450\sqrt{\sin(0.62t)}=400$

vehicles $=450\sqrt{\sin(0.62t)}$

$\sqrt{\sin(0.62t)}=\frac{8}{9}$

if $\sqrt{\sin(0.62t)}=1$,

$\sin(0.62t)=\frac{64}{81}$

vehicles $=450$

$t=\dfrac{\arcsin\left(\frac{64}{81}\right)}{0.62}$ $t=1.469\ldots$

$t\to B$

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

Answer QUESTION 1 parts (a) and (b) on this page.

Response for question 1(a)

$\int_1^5 A(t)\,dx$

$\int_1^5 450\sqrt{\sin(.62t)}\,dt$

Response for question 1(b)

~~$\frac{A(5) - A(1)}{5-1} =$~~

$\frac{1}{5}\int_1^5 A'(t)\,dt$

81.0498

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Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

increasing because $A'(t)$ is positive @ t=1 ∴ the rate would be increasing

$A'(1) = 148.447$

Response for question 1(d)

$0 = \int_0^t (A(x) - 400)\,dx$

Question 1

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

The context of this problem is vehicles arriving at a toll plaza at a rate of $A(t) = 450\sqrt{\sin(0.62t)}$ vehicles per hour, with time t measured in hours after 5 A.M., when there are no vehicles in line.

In part (a) students were asked to write an integral expression that gives the total number of vehicles that arrive at the plaza from time $t = 1$ to time $t = 5$. A correct response would report $\int_1^5 A(t)\,dt$.

In part (b) students were asked to find the average value of the rate of vehicles arriving at the toll plaza over the same time interval, $t = 1$ to $t = 5$. A correct response would report $\frac{1}{4}\int_1^5 A(t)\,dt$ and then evaluate this definite integral using a calculator to find an average value of 375.537. (The units, vehicles per hour, were given in the statement of the problem.)

In part (c) students were asked to reason whether the rate of vehicles arriving at the toll plaza is increasing or decreasing at 6 A.M., when $t = 1$. A correct response would use a calculator to determine that $A'(1)$, the derivative of the function $A(t)$ at this time, is positive ($A'(1) = 148.947$) and conclude that because $A'(1)$ is positive, the rate of vehicles arriving at the toll plaza is increasing.

Finally, in part (d) students were told that a line of vehicles forms when $A(t) \geq 400$ and the number of vehicles in line is given by the function $N(t) = \int_a^t (A(x) - 400)\,dx$, where a denotes the time, $a \leq t \leq 4$, when the line first begins to form. Students were asked to find the greatest number of vehicles in line at the plaza, to the nearest whole number, in the time interval $a \leq t \leq 4$ and to justify their answer. A correct response would recognize that the greatest number of vehicles is the maximum value of $N(t)$ on the closed interval $a \leq t \leq 4$. To find this maximum, a response should first determine the times t, $0 < t \leq 4$, when the derivative of $N(t)$ is 0. This requires using the Fundamental Theorem of Calculus to find $N'(t) = A(t) - 400$ and then using a calculator to determine that $N'(t)$ is equal to zero when $t = a = 1.469372$ and when $t = b = 3.597713$. A response should then evaluate the function $N(t)$ at each of the values $t = a$, $t = b$, and $t = 4$ to determine that the greatest number of vehicles in line is $N(b) = 71$.

Sample: 1A
Score: 9

The response earned 9 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 4 points in part (d).

In part (a) the response earned the point with the definite integral presented.

In part (b) the response earned the first point for the average value expression on the left side of the given equation. The second point was earned for the number on the right side of the equation, which is correct to three decimal places.

In part (c) the response earned the first point for the left side of the equation in the first line. The second point was earned with the concluding sentence.

Question 1 (continued)

In part (d) the response earned the first point with the equation on the left side of line 1. The middle expression, "$A(t) - 400$," of the equation is not needed to earn that point. The second point was earned with the values at the end of line 1 and line 2. The third point was earned in line 2 of the sentence on the right by identifying 71. The Candidates Test table on the left side earned the fourth point. The sentence on the right is consistent with the information in the Candidates Test table.

Sample: 1B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the point with the definite integral presented in line 1.

In part (b) the response earned the first point for the expression in line 1. The second point was earned for the correct value in line 2.

In part (c) the response earned the first point for the left side of the equation in line 2. The second point was earned with the concluding sentence.

In part (d) the response earned the first point with the equation on the left in line 3. The second point was not earned because the value of 3.598 is never given. The third point was not earned because there is no value given for $N(3.598)$. The fourth point was not earned because no justification is presented.

Sample: 1C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d).

In part (a) the response earned the point with the definite integral presented in line 1. The definite integral in line 2 is not necessary.

In part (b) the response did not earn the first point because the integrand given in line 1 after the crossed-out work is $A'(t)$ and not $A(t)$. Because the integrand presented is $A'(t)$ the response is not eligible to earn the second point.

In part (c) the response earned the first point with the statement "$A'(t)$ is positive @ $t = 1$" in line 1. The second point was earned in line 1 with the prior words: "increasing because $A'(t)$ is positive @ $t = 1$."

In part (d) the response did not earn any points because no correct work is presented.

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Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 2

☑ Scoring Guidelines

☑ Student Samples

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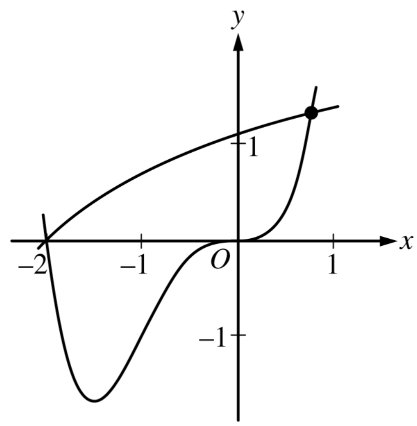
Part A (AB): Graphing calculator required
Question 2 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Let f and g be the functions defined by $f(x) = \ln(x+3)$ and $g(x) = x^4 + 2x^3$. The graphs of f and g, shown in the figure above, intersect at $x = -2$ and $x = B$, where $B > 0$.

	Model Solution	Scoring	
(a)	Find the area of the region enclosed by the graphs of f and g.		
	$\ln(x+3) = x^4 + 2x^3 \Rightarrow x = -2,\ x = B = 0.781975$	Integrand	**1 point**
	$\int_{-2}^{B} (f(x) - g(x))\,dx = 3.603548$	Limits of integration	**1 point**
	The area of the region is 3.604 (or 3.603).	Answer	**1 point**

Scoring notes:

- Other forms of the integrand in a definite integral, e.g., $|f(x) - g(x)|$, $|g(x) - f(x)|$, or $g(x) - f(x)$, earn the first point.
- To earn the second point, the response must have a lower limit of -2 and an upper limit expressed as either the letter B with no value attached, or a number that is correct to the number of digits presented, with at least one and up to three decimal places.
  - Case 1: If the response did not earn the second point because of an incorrect value of B, $0 < B < 1$, but used a lower limit of -2, the response earns the third point only for a consistent answer.
  - Case 2: If the response did not earn the second point because the lower limit used was $x = 0$, but the response used a correct upper limit of B, the response earns the third point for a consistent answer of 0.708 (or 0.707).
  - Case 3: If a response uses any other incorrect limits it does not earn the second or third points.
- A response containing the integrand $g(x) - f(x)$ must interpret the value of the resulting integral correctly to earn the third point. For example, the following response earns all 3 points: $\int_{-2}^{B}(g(x) - f(x))\,dx = -3.604$ so the area is 3.604. However, the response "Area $= \int_{-2}^{B}(g(x) - f(x))\,dx = 3.604$" presents an untrue statement and earns the first and second points but not the third point.
- A response must earn the first point in order to be eligible for the third point. If the response has earned the second point, then only the correct answer will earn the third point.
- Instructions for scoring a response that presents an integrand of $\ln(x + 3) - x^4 + 2x^3$ and the correct answer are shown in the "Global Special Case" after part (d).

Total for part (a) 3 points

(b) For $-2 \le x \le B$, let $h(x)$ be the vertical distance between the graphs of f and g. Is h increasing or decreasing at $x = -0.5$? Give a reason for your answer.

$h(x) = f(x) - g(x)$ $h'(x) = f'(x) - g'(x)$ $h'(-0.5) = f'(-0.5) - g'(-0.5) = -0.6$ (or -0.599)	Considers $h'(-0.5)$ – OR – $f'(x) - g'(x)$	**1 point**
Since $h'(-0.5) < 0$, h is decreasing at $x = -0.5$.	Answer with reason	**1 point**

Scoring notes:

- The response need not present the value of $h'(-0.5)$. The last line earns both points. However, if a value is presented it must be correct for the digits reported up to three decimal places.
- A response that reports an incorrect value of $h'(-0.5)$ earns only the first point.
- A response that presents only $h'(x)$ does not earn either point.
- The only response that earns the second point for concluding "h is increasing" is described in the "Global Special Case" provided after part (d).
- A response that compares the values of $f'(x)$ and $g'(x)$ at $x = -0.5$ earns the first point and is eligible for the second point. This comparison can be made symbolically or verbally; for example, the response "the rate of change of $f(x)$ is less than the rate of change of $g(x)$ at $x = -0.5$" earns the first point.

Total for part (b) 2 points

(c) The region enclosed by the graphs of f and g is the base of a solid. Cross sections of the solid taken perpendicular to the x-axis are squares. Find the volume of the solid.

$\int_{-2}^{B} (f(x) - g(x))^2 \, dx = 5.340102$	Integrand **1 point**
The volume of the solid is $5.340.$	Answer **1 point**

Scoring notes:

- The first point is earned for an integrand of $k(f(x) - g(x))^2$ or its equivalent with $k \neq 0$ in any definite integral. If $k \neq 1,$ then the response is not eligible for the second point.
- A response that does not earn the first point is ineligible to earn the second point, with the following exceptions:
  - A response which has a presentation error in the integrand (for example, mismatched or missing parentheses, misplaced exponent) does not earn the first point but would earn the second point for the correct answer. A response which has a presentation error in the integrand and which reports an incorrect answer earns no points.
  - A response that presents an integrand of $\left(\ln(x + 3) - x^4 + 2x^3\right)^2$. Scoring instructions for this case are provided in the "Global Special Case" after part (d).
- A response that uses incorrect limits is only eligible for the second point, provided the limits are imported from part (a) in Case 1 or Case 2. In both of these situations, the second point is earned only for answers consistent with the imported limits.

Total for part (c) 2 points

(d) A vertical line in the xy-plane travels from left to right along the base of the solid described in part (c). The vertical line is moving at a constant rate of 7 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5.$

The cross section has area $A(x) = (f(x) - g(x))^2.$ $\frac{d}{dt}[A(x)] = \frac{dA}{dx} \cdot \frac{dx}{dt}$	$\frac{dA}{dx} \cdot \frac{dx}{dt}$ **1 point**
$\frac{d}{dt}[A(x)]\Big\vert_{x=-0.5} = A'(-0.5) \cdot 7 = -9.271842$ At $x = -0.5,$ the area of the cross section above the line is changing at a rate of -9.272 (or -9.271) square units per second.	Answer **1 point**

Scoring notes:

- The first point may be earned by presenting $\frac{dA}{dx} \cdot \frac{dx}{dt}$, $A' \cdot \frac{dx}{dt}$, $A'(x) \cdot \frac{dx}{dt}$, $A'(x) \cdot x'$, $A'(-0.5) \cdot 7$, or $k \cdot 7$, where k is a declared value of $A'(-0.5)$, or any equivalent expression, including $2(f(x) - g(x))(f'(x) - g'(x))\frac{dx}{dt}$.
- If a response defines $f(x) - g(x)$ as a function in parts (b) or (c) (for example, $h(x) = f(x) - g(x)$), then a correct expression for $\frac{dA}{dt}$ (for example, $2h\frac{dh}{dt}$) earns the first point.
- A response that imports a function $A(x)$ declared in part (c) is eligible for both points (the answer must be consistent with the imported function $A(x)$).
- A response that presents an incorrect function for $A(x)$ that is not imported from part (c) is eligible only for the first point.
- Except when $A(x)$ is imported from part (c), the second point is earned only for the correct answer.
- A response that does not earn the first point is ineligible to earn the second point except in the special case noted below.

Total for part (d) 2 points

Total for question 2 9 points

Global Special Case: A response may incorrectly simplify $f(x) - g(x)$ to $j(x) = \ln(x + 3) - x^4 + 2x^3$ instead of $\ln(x + 3) - x^4 - 2x^3$. Because this question is calculator active, a response with this incorrect simplification may nevertheless present correct answers.

- In any part of the question, a response that starts correctly by using $f(x) - g(x)$, then presents $j(x)$, is eligible for all points in that part.
- The first time a response implicitly presents $f(x) - g(x)$ as $j(x) = \ln(x + 3) - x^4 + 2x^3$ (with no explicit connection) in any part of this question, the response loses a point. The response is then eligible for all remaining points for a correct or consistent answer.
- In part (a) the consistent answer using $j(x)$ is negative and will not earn the third point.
- In part (b) the consistent answer using $j(x)$ is that $j'(-0.5) = 2.4 > 0$, so h is increasing at $x = -0.5$.
- In part (c) the consistent answer using $j(x)$ is 252.187 (or 252.188).
- In part (d) the consistent answer using $j(x)$ is 20.287.

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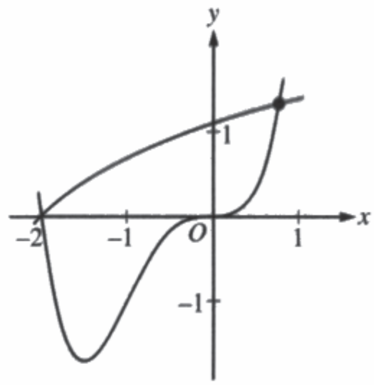
Answer QUESTION 2 parts (a) and (b) on this page.



Response for question 2(a)

B= .7819751

$$\int_{-2}^{.7819751} \left(\ln(x+3) - (x^4+2x^3)\right)dx = 3.604$$

Response for question 2(b)

$h(x) = \ln(x+3) - (x^4 + 2x^3)$

h(x) is decreasing at x=-.5 because $h'(x) < 0$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5105/6

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Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$A = s^2 \quad s = h(x)$

$V = \int_{-2}^{B} (h(x))^2\,dx = 5.340$

Response for question 2(d)

$x = -.5$

$A = (h(x))^2$

$\frac{dx}{dt} = 7$ u/s

$\frac{dA}{dt} = 2h(x) \cdot h'(x) \cdot \frac{dx}{dt}$

$\frac{dA}{dt} = 2(1.10379) \cdot (-.6) \cdot 7 = -9.272$ units^2/second

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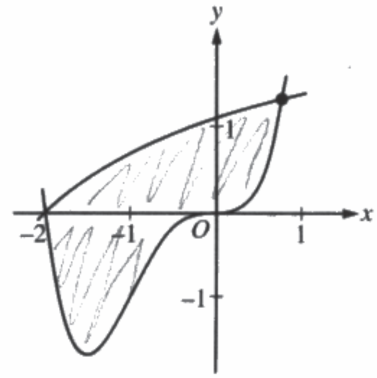
Answer QUESTION 2 parts (a) and (b) on this page.



Response for question 2(a)

B = 0.78198

$$\int_{-2}^{0.78198} (f(x) - g(x))\,dx = 3.6035$$

Response for question 2(b)

$$\int_{-2}^{-0.5} (f(x) - g(x))\,dx = 2.3657$$

$$\int_{-0.5}^{B} (f(x) - g(x))\,dx = 1.2378$$

Decreasing because the area between f(x) and g(x) is greater from ~~x = −2 to~~ x = −0.5 than from x = −0.5 to x = B.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5108/6

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$$\int_{-2}^{B} (f(x)-g(x))^2 \, dx$$

↓

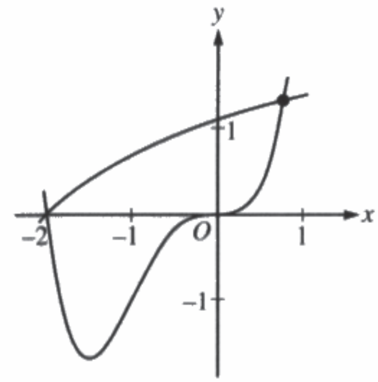
5.3401

Response for question 2(d)

$A = (f(x) - g(x))$

$\frac{dA}{dt} = (f' \frac{dx}{dt} - g' \frac{dx}{dt})$

Answer QUESTION 2 parts (a) and (b) on this page.



Response for question 2(a)

$\int_{-2}^{.782} [f(x)-g(x)]\,dx = 12.022$

Response for question 2(b)

increasing at t = −0.5 b/c y = .916 and is both positive and above the x axis, so therefore its increasing on t = −0.5

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$\pi \int_{-2}^{.782} [f(x) - g(x)]^2 \, dx = 792.272$ units^3

Response for question 2(d)

$A(t)/dt = 1.104$ units per second

$A'(t) = f(x) - g(x)$

$A'(-.5) = f(-.5) - g(-.5)$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0009240

Q5108/7

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem students were provided graphs of the functions $f(x) = \ln(x + 3)$ and $g(x) = x^4 + 2x^3$ and told that the graphs intersect at $x = -2$ and $x = B$, where $B > 0$.

In part (a) students were asked to find the area of the region enclosed by the graphs of f and g. A correct response provides the setup of the definite integral of $f(x) - g(x)$ from $x = -2$ to $x = B$. The response must determine the value of B (although this value need not be presented) and then use this value to evaluate the integral and find an area of 3.604.

In part (b) the function $h(x)$ is defined to be the vertical distance between the graphs of f and g, and students were asked to reason whether h is increasing or decreasing at $x = -0.5$. A correct response would recognize that the vertical distance between the graphs of f and g is $f(x) - g(x)$ and then evaluate the derivative $h'(x) = f'(x) - g'(x)$ at $x = -0.5$. Because this value is negative, the response should conclude that h is decreasing when $x = -0.5$.

In part (c) students were told that the region enclosed by the graphs of f and g is the base of a solid with cross sections of the solid taken perpendicular to the x-axis that are squares. Students were asked to find the volume of the solid. A correct response would realize that the area of a cross section is $(f(x) - g(x))^2$ and would find the requested volume by integrating this area from $x = -2$ to $x = B$.

In part (d) students were told that a vertical line in the xy-plane travels from left to right along the base of the solid described in part (c) at a constant rate of 7 units per second. Students were asked to find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$. A correct response would again use the area function from part (c), $A(x) = (f(x) - g(x))^2$ and the chain rule to find $\frac{d}{dt}[A(x)] = \frac{dA}{dx} \cdot \frac{dx}{dt}$. The response should then use a calculator to find $A'(-0.5)$ and multiply this value by the given value of $\frac{dx}{dt} = 7$.

Sample: 2A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d).

In part (a) the response earned the first point with the correct integrand in a definite integral. The response earned the second point with correct limits on the definite integral. The response earned the third point with the circled correct answer.

In part (b) the response earned the first point by noting that $x = -0.5$ in the second line and stating that $h'(x) < 0$ in the third line. The response earned the second point with a correct answer in the second line and by stating that $h'(x) < 0$ at $x = -0.5$ in the second and third lines.

Question 2 (continued)

In part (c) the response earned the first point with the correct integrand in the definite integral. The function $h(x)$ is defined in part (b). The response is eligible for the second point because the limits of integration are -2 and B, for B defined in part (a). The response earned the second point with the circled correct answer.

In part (d) the response earned the first point with the expression $2h(x) \cdot h'(x) \cdot \frac{dx}{dt}$ in the second line on the left, which is equivalent to $\frac{dA}{dx} \cdot \frac{dx}{dt}$. The function $h(x)$ is defined in part (b). The response would have earned the second point with "$2(1.10379) \cdot (-.6) \cdot 7$" in the middle of the third line on the left but went on to simplify and earned the point with the correct circled answer on the same line. The response provides correct units, but these are not required to earn the second point.

Sample: 2B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the first point with the correct integrand in a definite integral. The response earned the second point with correct limits on the definite integral. The response earned the third point with the correct circled answer.

In part (b) the response did not earn the first point because it considers neither $f'(x) - g'(x)$ nor $h'(-0.5)$. Because the response did not earn the first point, it is not eligible for the second point.

In part (c) the response earned the first point with the correct integrand in the definite integral. The response earned the second point with the correct answer boxed in line 2.

In part (d) the response is eligible for the first point because the incorrect expression for A is a nonconstant function. The response earned the first point in the second line by expressing $\frac{dA}{dt}$ in a form equivalent to $\frac{dA}{dx} \cdot \frac{dx}{dt}$. The response is not eligible for the second point because the expression for A is incorrect.

Sample: 2C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response earned the first point with a correct integrand in a definite integral. The response earned the second point with correct limits on the definite integral. The response did not earn the third point because the answer is incorrect.

In part (b) the response did not earn the first point because it considers neither $f'(x) - g'(x)$ nor $h'(-0.5)$. Because the response did not earn the first point, it is not eligible for the second point.

In part (c) the response earned the first point with an integrand of the form $k(f(x) - g(x))^2$ with $k \neq 0$ in a definite integral. Because $k \neq 1$, the response is not eligible for the second point.

Question 2 (continued)

In part (d) the response did not earn the first point because the expression for $\frac{dA}{dt}$ in the second line is not of the form $\frac{dA}{dx} \cdot \frac{dx}{dt}$. Because the response did not earn the first point, it is not eligible for the second point.

2022

AP CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 3

☑ Scoring Guidelines

☑ Student Samples

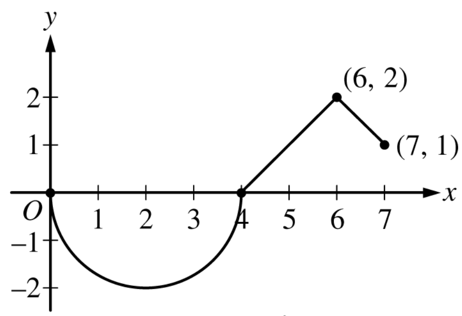
☑ Scoring Commentary

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Graph of f'

Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

	Model Solution	Scoring	
(a)	Find $f(0)$ and $f(5)$.		
	$f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 + 2\pi$ $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$	Area of either region – OR – $\int_0^4 f'(x)\,dx$ – OR – $\int_4^5 f'(x)\,dx$	**1 point**
		$f(0)$	**1 point**
		$f(5)$	**1 point**

Scoring notes:

- A response with answers of only $f(0) = \pm 2\pi$, or only $f(5) = \frac{1}{2}$, or both earns 1 of the 3 points.
- A response displaying $f(5) = \frac{7}{2}$ and a missing or incorrect value for $f(0)$ earns 2 of the 3 points.
- The second and third points can be earned in either order.
- Read unlabeled values from left to right and from top to bottom as $f(0)$ and $f(5)$. A single value must be labeled as either $f(0)$ or $f(5)$ in order to earn any points.

Total for part (a) — 3 points

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

The graph of f has a point of inflection at each of $x = 2$ and $x = 6$, because $f'(x)$ changes from decreasing to increasing at $x = 2$ and from increasing to decreasing at $x = 6$.	Answer	**1 point**
	Justification	**1 point**

Scoring notes:

- A response that gives only one of $x = 2$ or $x = 6$, along with a correct justification, earns 1 of the 2 points.
- A response that claims that there is a point of inflection at any value other than $x = 2$ or $x = 6$ earns neither point.
- To earn the second point a response must use correct reasoning based on the graph of f'. Examples of correct reasoning include:
  - Correctly discussing the signs of the slopes of the graph of f'
  - Citing $x = 2$ and $x = 6$ as the locations of local extrema on the graph of f'
- Examples of reasoning not (sufficiently) connected to the graph of f' include:
  - Reasoning based on sign changes in f'' unless the connection is made between the sign of f'' and the slopes of the graph of f'
  - Reasoning based only on the concavity of the graph of f
- The second point cannot be earned by use of vague or undefined terms such as "it" or "the function" or "the derivative."
- Responses that report inflection points as ordered pairs must report the points $(2, 3 + \pi)$ and $(6, 5)$ in order to earn the first point. If the y-coordinates are reported incorrectly, the response remains eligible for the second point.

Total for part (b) 2 points

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

$g'(x) = f'(x) - 1$	$g'(x) = f'(x) - 1$	**1 point**
$f'(x) - 1 \le 0 \Rightarrow f'(x) \le 1$ The graph of g is decreasing on the interval $0 \le x \le 5$ because $g'(x) \le 0$ on this interval.	Interval with reason	**1 point**

Scoring notes:

- The first point can be earned for $f'(x) \le 1$ or the equivalent, in words or symbols.
- Endpoints do not need to be included in the interval to be eligible for the second point.

Total for part (c) 2 points

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

g is continuous, $g'(x) < 0$ for $0 < x < 5$, and $g'(x) > 0$ for $5 < x < 7$. Therefore, the absolute minimum occurs at $x = 5$, and $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$ is the absolute minimum value of g.	Considers $g'(x) = 0$	**1 point**
	Answer with justification	**1 point**

Scoring notes:

- A justification that uses a local argument, such as "g' changes from negative to positive (or g changes from decreasing to increasing) at $x = 5$" must also state that $x = 5$ is the only critical point.
- If $g'(x) = 0$ (or equivalent) is not declared explicitly, a response that isolates $x = 5$ as the only critical number belonging to $(0, 7)$ earns the first point.
- A response that imports $g'(x) = f'(x)$ from part (c) is eligible for the first point but not the second.
  - In this case, consideration of $x = 4$ as the only critical number on $(0, 7)$ earns the first point.
- Solution using Candidates Test:

  $g'(x) = f'(x) - 1 = 0 \Rightarrow x = 5,\ x = 7$

x	$g(x)$
0	$3 + 2\pi$
5	$-\frac{3}{2}$
7	$-\frac{1}{2}$

  The absolute minimum value of g on the interval $0 \le x \le 7$ is $-\frac{3}{2}$.

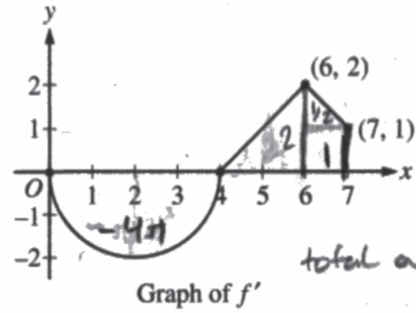
- When using a Candidates Test, a response may import an incorrect value of $f(0) = g(0) > -\frac{3}{2}$ from part (a). The second point can only be earned for an answer of $-\frac{3}{2}$.

Total for part (d) 2 points

Total for question 3 9 points

3 3 3 3 3 **NO CALCULATOR ALLOWED** 3 3 3 3 3

Answer QUESTION 3 parts (a) and (b) on this page.



Graph of f'

$f(4)=3$

total area: $3.5-4\pi$

Response for question 3(a)

find $f(0)$ & $f(5)$

$\int_4^0 f'(x)\,dx = f(0)-f(4)$

$-\int_0^4 f'(x)\,dx = f(0)-f(4)$

$4\pi = f(0)-3$

$f(0)=4\pi+3$

$\int_4^5 f'(x)\,dx = f(5)-f(4)$

$\frac{1}{2} = f(5)-3$

$f(5)=3.5$

Response for question 3(b)

POI: $x=2$, $x=6$

a point of inflection occurs when the second derivative changes sign. The graph is a graph of f', ~~and f'' is the slope~~ of f', so wherever the graph changes from increasing to decreasing, or ~~the~~ vice versa is a point of inflection

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5105/8

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

$g'(x) = f'(x) - 1$

$f'(x) - 1 = 0$

$f'(x) = 1$

$x = 5 \quad x = 7$

decreasing in interval (0, 5), because $g'(x) = f'(x) - 1$ is negative in this interval, meaning g is decreasing.

Response for question 3(d)

crit points: $x = 5 \quad x = 7$
endpoints: $x = 0 \quad x = 7$

$g(0) = f(0) - 0 = 4\pi + 3$

$g(5) = f(5) - 5 = 3.5 - 5 = -1.5$

$g(7) = f(7) - 7 = 6.5 - 7 = -0.5$

absolute minimum: **(5, −1.5)**

$\int_4^7 f'(x)\,dx = f(7) - f(4)$

$3.5 = f(7) - f(4)$

$3.5 = f(7) - 3$

$f(7) = 6.5$

The abs min of g(x) is −1.5 based on the values found through the critical points of the function

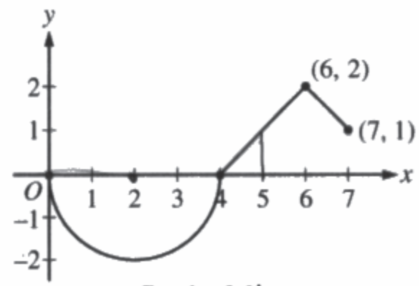
Page 9

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5106/9

3 3 3 3 3 **NO CALCULATOR ALLOWED** 3 3 3 3 3

Answer QUESTION 3 parts (a) and (b) on this page.



Graph of f'

Response for question 3(a)

$f(4)=3$

using FTC:

$\int_0^4 f'(x) = f(4) - f(0)$

$\int_0^4 f'(x) = \frac{1}{2}\pi(2)^2 = -2\pi$

$-2\pi = 3 - f(0)$

$\boxed{f(0) = 3 + 2\pi}$

$\int_4^5 f'(x) = f(5) - f(4)$

$\int_4^5 f'(x) = \frac{1}{2}(1\times 1) = \frac{1}{2}$

$\frac{1}{2} = f(5) - 3$

$\boxed{f(5) = 7/2}$

Response for question 3(b)

point of inflection is where $f''(x) = 0$ (horizontal tangent)

There is a horizontal tangent at the bottom of the semicircle, at $\boxed{x = 2}$.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5106/8

3 3 3 3 3 **NO CALCULATOR ALLOWED** 3 3 3 3 3

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

$g(x) = f(x) - x$

$g'(x) = f'(x) - 1$

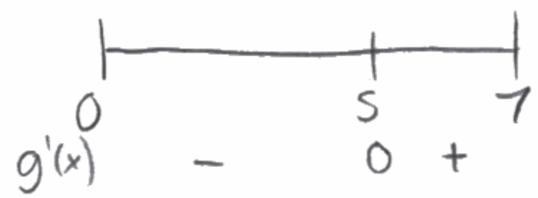
g is decreasing when $g'(x) < 0$

$g'(x) < 0$ when $f'(x) < 1$

$f'(x) < 1$ on the interval $\boxed{0 < x < 5}$

Response for question 3(d)

minimum is where the first derivative goes from negative to positive.



$g'(x)$ goes from negative to positive at $x = 5$

$g(5) = f(5) - (5)$ $g(5) = 7/2 - 5 = \boxed{-3/2 \text{ units}}$

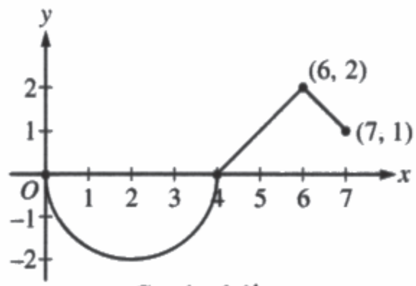
Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5106/9

Sample 3C

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (a) and (b) on this page.



Graph of f'

Response for question 3(a)

$f(0) = \int_0^0 f'(x) = 0$ $\quad f(0) = 0$

$f(5) = \int_0^5 f'(x) = \frac{1}{2} - 2\pi$ $\quad f(5) = \frac{1}{2} - 2\pi$

$-\frac{1}{2}\pi r^2$

$-\frac{1}{2} 4\pi$ $\quad -2\pi + \frac{1}{2}$

Response for question 3(b)

$x = 0$ At $x = 0$ and $x = 4$, the first derivative

$x = 4$ switches signs, meaning there is a zero, or

a point of inflection for the second

derivative.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5113/8

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

g is dec on the intervals (0,7) this is because g is negative along the whole graph, making it decreasing on the whole graph.

Response for question 3(d)

$g'(x) = f'(x) - 1$

$f'(x) - 1 = 0$

$x = 5$

abs min at $x = 5$

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the graph of a function f', which consists of a semicircle and two line segments on the interval $0 \le x \le 7$, is provided. It is also given that this is the graph of the derivative of a differentiable function f with $f(4) = 3$.

In part (a) students were asked to find $f(0)$ and $f(5)$. To find $f(0)$ a correct response uses geometry and the Fundamental Theorem of Calculus to calculate the signed area of the semicircle, $\int_0^4 f'(x)\,dx = -2\pi$, and subtracts this value from the initial condition, $f(4) = 3$, to obtain a value of $3 + 2\pi$. To find $f(5)$ a correct response would add the initial condition to the signed area $\int_4^5 f'(x)dx = \frac{1}{2}$, found using geometry, to obtain a value of $\frac{7}{2}$.

In part (b) students were asked to find the x-coordinates of all points of inflection on the graph of f for $0 < x < 7$ and to justify their answers. A correct response would use the given graph to determine that the graph of $f'(x)$ changes from decreasing to increasing, or vice versa, at the points $x = 2$ and $x = 6$. Therefore, these are the inflection points of the graph of f.

In part (c) students were told that $g(x) = f(x) - x$ and are asked to determine on which intervals, if any, the function g is decreasing. A correct response would find that $g'(x) = f'(x) - 1$ and then use the given graph of f' to determine that when $0 \le x \le 5$, $f'(x) \le 1 \Rightarrow g'(x) \le 0$. Therefore, g is decreasing on the interval $0 \le x \le 5$.

In part (d) students were asked to find the absolute minimum value of $g(x) = f(x) - x$ on the interval $0 \le x \le 7$. A correct response would use the work from part (c) to conclude $g'(x) < 0$ for $0 < x < 5$ and $g'(x) > 0$ for $5 < x < 7$. Thus the absolute minimum of g occurs at $x = 5$. Using the work from part (a), which found the value of $f(5)$, the absolute minimum value of g is $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$.

Sample: 3A
Score: 8

The response earned 8 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d).

In part (a) the first point was earned for the integral expression to the left of the first equal sign. The second point was not earned because the value of $f(0)$ is reported incorrectly. The third point was earned because the stated value of $f(5)$ is correct.

In part (b) the first point was earned for the boxed answer on lines 1 and 2, which declares the "POI" as $x = 2$ and $x = 6$. The second point was earned for the explanation given in the paragraph below the boxed answer. The response appeals to the change in sign of f'', which alone would not be sufficient but goes on to declare that "f'' is the slope of f'" and indicates correctly that where the graph (of f') "changes from increasing to decreasing, or vice versa is a point of inflection."

Question 3 (continued)

In part (c) the first point was earned on line 1 for $g'(x) = f'(x) - 1$. The second point was earned on lines 1, 2, and 3 on the right for giving the correct interval $(0, 5)$ with the reason that "$g'(x) = f'(x) - 1$ is negative in this interval."

In part (d) the first point was earned on line 1 for consideration of only $x = 5$ (and the endpoints) as possible locations of the absolute minimum. The second point was earned on line 8 for declaring -1.5 as the absolute minimum. A Candidates Test is carried out correctly, with an incorrect value of $f(0)$ that is greater than $-\frac{3}{2}$ imported from part (a). Such responses are still eligible to earn the second point as long as $g(0)$ is declared to be that same imported value, and there are no mistakes in reported values of $g(5)$ and $g(7)$.

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the first point for $\int_0^4 f'(x)$ on line 3 on the left. Note that the integrals are missing the differential dx; this oversight is not penalized. The second and third points were earned for the boxed, correct values of $f(0)$ and $f(5)$.

In part (b) the response did not earn the first point, because $x = 6$ is not given among the answers. The second point was not earned because the reasoning that "there is a horizontal tangent at the bottom of the semicircle" is not sufficient. A response must make a specific appeal to f' in order for the second point to be earned.

In part (c) the response earned the first point on line 2 for the correct derivative of $g'(x)$. The response earned the second point for the correct boxed interval on line 5, with correct reasoning on lines 3 and 4.

In part (d) the response earned the first point for consideration of a sign change in $g'(x)$ below the sign chart. Although the response does have the correct answer of $-\frac{3}{2}$, it uses the local argument that "$g'(x)$ goes from negative to positive at $x = 5$" without an appeal to the whole interval $(0, 7)$. Therefore, the second point was not earned. The response appears, perhaps, to address the interval with a sign chart, but the response must explain the conclusions gathered from the chart in order for the point to be earned.

Sample: 3C
Score: 2

The response earned 2 points: 1 point in part (a), no points in part (b), no points in part (c), and 1 point in part (d).

In part (a) the response earned the first point on line 2 for consideration of $\frac{1}{2}$ (the area of the necessary triangular region) to the right of the second equal sign. The response did not earn the second and third points because the answers for $f(0)$ and $f(5)$ are both incorrect.

In part (b) the response did not earn either point because answers other than $x = 2$ or $x = 6$ are given (in this case, $x = 0$ and $x = 4$), which renders the response ineligible for either point.

Question 3 (continued)

In part (c) the response did not earn the first point because $g'(x)$ (or equivalent) is not considered. It is not possible to earn the second point without having earned the first point in part (c).

In part (d) the response earned the first point on line 2 for "$f'(x) - 1 = 0.$" No function value at $x = 5$ is reported; thus, the second point was not earned.

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Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 4

☑ Scoring Guidelines

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Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

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Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	−6.1	−5.0	−4.4	−3.8	−3.5

An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

Model Solution	Scoring	

(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

$r''(8.5) \approx \dfrac{r'(10) - r'(7)}{10 - 7} = \dfrac{-3.8 - (-4.4)}{10 - 7}$	$r''(8.5)$ with supporting work	**1 point**
$= \dfrac{0.6}{3} = 0.2$ centimeter per day per day	Units	**1 point**

Scoring notes:

- To earn the first point the supporting work must include at least a difference and a quotient.
- Simplification is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The second point can be earned with an incorrect approximation for $r''(8.5)$ but cannot be earned without some value for $r''(8.5)$ presented.
- Units may be written in any equivalent form (such as cm/day^2).

Total for part (a) — 2 points

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

$r(t)$ is twice-differentiable. $\Rightarrow$ $r'(t)$ is differentiable. $\Rightarrow$ $r'(t)$ is continuous. $r'(0) = -6.1 < -6 < -5.0 = r'(3)$ Therefore, by the Intermediate Value Theorem, there is a time t, $0 < t < 3$, such that $r'(t) = -6$.	$r'(0) < -6 < r'(3)$	**1 point**
	Conclusion using Intermediate Value Theorem	**1 point**

Scoring notes:

- To earn the first point, the response must establish that -6 is between $r'(0)$ and $r'(3)$ (or -6.1 and -5). This statement may be represented symbolically (with or without including one or both endpoints in an inequality) or verbally. A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not state that -6 is between -6.1 and -5. Thus this response does not earn the first point.
- To earn the second point:
  - The response must state that $r'(t)$ is continuous because $r'(t)$ is differentiable (or because $r(t)$ is twice differentiable).
  - The response must have earned the first point.
    - Exception: A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not earn the first point because of imprecise communication but may nonetheless earn the second point if all other criteria for the second point are met.
  - The response must conclude that there is a time t such that $r'(t) = -6$. (A statement of "yes" would be sufficient.)
- To earn the second point, the response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

Total for part (b) 2 points

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

$\int_0^{12} r'(t)\,dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12)$	Form of right Riemann sum	**1 point**
$= 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ $= -51$	Answer	**1 point**

Scoring notes:

- To earn the first point, at least seven of the eight factors in the Riemann sum must be correct. If there is any error in the Riemann sum, the response does not earn the second point.
- A response of $3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ earns both the first and second points, unless there is a subsequent error in simplification, in which case the response would earn only the first point.
- A response that presents the correct answer, with accompanying work that shows the four products in the Riemann sum (without explicitly showing all of the factors and/or the sum process) does not earn the first point but earns the second point. For example, $-15 + 4(-4.4) + 3(-3.8) + -7$ does not earn the first point but earns the second point. Similarly, $-15,\ -17.6,\ -11.4,\ -7 \rightarrow -51$ does not earn the first point but earns the second point.
- A response that presents the correct answer (-51) with no supporting work earns no points.
- A response that provides a completely correct left Riemann sum and approximation $\int_0^{12} r'(t)\,dt$ (i.e., $3r'(0) + 4r'(3) + 3r'(7) + 2r'(10) = 3(-6.1) + 4(-5.0) + 3(-4.4) + 2(-3.8) = -59.1$ earns 1 of the 2 points. A response that has any error in a left Riemann sum or evaluation for $\int_0^{12} r'(t)\,dt$ earns no points.
- Units are not required or read in this part.

Total for part (c) 2 points

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

$\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}$	Product rule	**1 point**
	Chain rule	**1 point**
$\left.\frac{dV}{dt}\right\vert_{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2) = -\frac{70{,}000\pi}{3}$ The rate of change of the volume of the sculpture at $t = 3$ is approximately $-\frac{70{,}000\pi}{3}$ cubic centimeters per day.	Answer	**1 point**

Scoring notes:

- The first 2 points could be earned in either order.
- A response with a completely correct product rule, missing one or both of the correct differentials, earns the product rule point, but not the chain rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh + \frac{1}{3}\pi r^2$ earns the first point, but not the second.
- A response that treats r or h (but not both) as a constant is eligible for the chain rule point but not the product rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}$ is correct if h is constant, and thus earns the chain rule point.
- Note: Neither $\frac{dV}{dt} = \frac{2}{3}\pi r\frac{dh}{dt}$ nor $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}\frac{dh}{dt}$ earns any points.
- A response that assumes a functional relationship between r and h (such as $r = 2h$), and uses this relationship to create a function for volume in terms of one variable, is eligible for at most the chain rule point. For example, $r = 2h \rightarrow V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3 \rightarrow \frac{dV}{dt} = 4\pi h^2\frac{dh}{dt}$ earns only the chain rule point.
- A response that mishandles the constant $\frac{1}{3}\pi$ cannot earn the third point but is eligible for the first 2 points.
- The third point cannot be earned without both of the first 2 points.
- $\frac{dV}{dt} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2)$ earns all 3 points.
- Units are not required or read in this part.

Total for part (d) 3 points

Total for question 4 9 points

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	−6.1	−5.0	−4.4	−3.8	−3.5

Response for question 4(a)

$$r''(8.5) \approx \frac{r'(10) - r'(7)}{10-7} = \frac{-3.8 - (-4.4)}{3} = \frac{-3.8+4.4}{3} = \frac{0.6}{3}$$

$= \boxed{0.2 \text{ cm/day}^2}$

Response for question 4(b)

Since r is twice-differentiable, r' is also differentiable on the same interval $0 \le t \le 12$. Therefore r' is differentiable between $0 \le t \le 3$ as well since that interval is inside the previous one.

If a function is differentiable, it is also continuous. Therefore, $r'(t)$ is also continuous on $0 \le t \le 3$. According to the Intermediate Value Theorem, any continuous function $f(x)$ on $[a,b]$ will take on every value between $f(a)$ and $f(b)$. Therefore $r'(t)$ will take on every value between −5.0 and −6.1. −6 is inside this range, so there is a time t where $r'(t) = -6$ on $0 \le t \le 3$.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5108/10

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$(3-0)(-5.0) + (7-3)(-4.4) + (10-7)(-3.8) + (12-10)(-3.5)$

$3(-5) + 4(-4.4) + 3(-3.8) + 2(-3.5)$

$-15 + -17.6 - 11.4 - 7$

$-15 - 29 - 7$

$-44 - 7$

$\boxed{-51}$

Response for question 4(d)

$\frac{dh}{dt} = -2 \text{ cm/day}$

$r(3) = 100 \text{ cm}$

$h(3) = 50 \text{ cm}$

$r'(3) = \frac{dr}{dt} = -5.0 \text{ cm/day}$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{1}{3}\pi\left[(r^2)\left[\frac{dh}{dt}\right] + (h)\left(2r\frac{dr}{dt}\right)\right]$

$= \frac{1}{3}\pi\left[(100^2)(-2) + (50)(2(100)(-5.0))\right]$

$= \frac{1}{3}\pi\left[-20000 + 50(-1000)\right]$

$= \frac{1}{3}\pi\left[-20000 + -50000\right]$

$= \boxed{\frac{-70000\pi}{3} \text{ cm}^3/\text{day}}$

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	−6.1	−5.0	−4.4	−3.8	−3.5

Response for question 4(a)

$$\begin{array}{r} 4.4 \\ 3.8 \\ \hline 0.6 \end{array}$$

$$\frac{-3.8 - -4.4}{10-7} = \frac{0.6}{3} = \frac{\frac{6}{10}}{\frac{3}{1}} = \frac{6}{10}\cdot\frac{1}{3} = \frac{6}{30} = \frac{1}{5}\ \text{centimeters}/\text{day}^2$$

Response for question 4(b)

$-5.0+6.1$

$$\frac{-5.0 - -6.1}{3-0} = \frac{1.1}{3}$$

$$\frac{\frac{11}{10}}{\frac{3}{1}} = \frac{11}{30}$$

NO, the Mean Value theorem does not garentee $r'(t)=-6$ on the interval $0\le t\le 3$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5113/10

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$3(-5.0)+4(-4.4)+3(-3.8)+2(-3.5)$

$-15-17.6-11.4+7 = \boxed{-37.0}$

Response for question 4(d)

$\frac{dh}{dt} = -2$ cm/day $t=3$ days

$r=100$cm

$V=\frac{1}{3}\pi r^2 h$ $h=50$cm

$\frac{dV}{dt} = \frac{2}{3}\pi r \frac{dr}{dt} \cdot h + \frac{dh}{dt} \cdot \frac{1}{3}\pi r^2$

$\frac{2}{3}\pi(100)$

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	−6.1	−5.0	−4.4	−3.8	−3.5

Response for question 4(a)

$$\frac{-3.8-(-4.4)}{10-7} = \frac{0.6}{3} = \boxed{0.2}$$

Response for question 4(b)

Yes, because the Mean Value Theorem states that if a function is continuous and differentiable, there will be a derived function that can find a value c if $a \le c \le b$. Since −6.0 is between −6.1 and −5.0 in the interval $0 \le t \le 3$, there will be a value for r where $r'(r) = 6$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5112/10

4 4 4 4 4 **NO CALCULATOR ALLOWED** 4 4 4 4 4

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$\int_0^{12} r'(t)\,dt = 2(-3.5) + 3(-3.8) + 4(-4.4) + 3(-5.0)$

$= -7.0 - 11.4 - 17.6 - 15.0 = \boxed{-52.0}$

Response for question 4(d)

$\frac{dh}{dt} = -2$

$r = 100$ cm

$h = 50$ cm

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{2}{3}\pi r \frac{dr}{dt} h \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{2}{3}\pi (100)(50)(-2)$

$\frac{dV}{dt} = \boxed{-\frac{20000}{3}\pi \ \frac{cm^3}{day}}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5112/11

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the melting of an ice sculpture can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is a twice-differentiable function $r(t)$ measured in centimeters, with time t, $0 \le t \le 12$, in days. Selected values of $r'(t)$ are provided in a table.

In part (a) students were asked to approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$ and to provide correct units. A correct response should estimate the value using a difference quotient, drawing from the data in the table that most tightly bounds $t = 8.5$. The response should include units of centimeters per day per day.

In part (b) students were asked to justify whether there is a time t, $0 \le t \le 3$, for which the rate of change of r is equal to -6. A correct response will use the Intermediate Value Theorem, first noting that the conditions for applying this theorem are met—specifically that $r'(t)$ is continuous because $r(t)$ is twice-differentiable and that -6 is bounded between the values of $r'(0)$ and $r'(3)$ given in the table. Therefore, by the Intermediate Value Theorem, there is a time t such that $0 < t < 3$, with $r'(t) = -6$.

In part (c) students were asked to use a right Riemann sum and the subintervals indicated by the table to approximate the value of $\int_0^{12} r'(t)\,dt$. A correct response should present the sum of the four products $\Delta t_i \cdot r'(t_i)$ drawn from the table and obtain an approximation value of -51.

In part (d) students were told that the height of the cone decreases at a rate of 2 centimeters per day and that at time $t = 3$ the radius of the cone is 100 cm and the height is 50 cm. They are asked to find the rate of change of the volume of the cone with respect to time at time $t = 3$ days. A correct response will use the product and chain rules to differentiate the given function for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, and then evaluate the resulting derivative using values $r = 100$, $h = 50$, $\left.\frac{dh}{dt}\right|_{t=3} = -2$, and $\left.\frac{dr}{dt}\right|_{t=3} = -5$ (from the table) to obtain a rate of $-\frac{70{,}000\pi}{3}$ cubic centimeters per day.

Sample: 4A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{-3.8-(-4.4)}{3}$ in line 1, with no simplification. In this case, correct simplification to the boxed answer of 0.2 in line 2 earned the first point. The response earned the second point for the correct units presented in the boxed answer in line 2.

Question 4 (continued)

In part (b) the response earned the first point with the statements given in the last three lines, which place the value -6 between the values of -5.0 and -6.1. The response earned the second point with the statements in lines 1 through 5, concluding that $r'(t)$ is continuous because $r'(t)$ is differentiable. The response names the Intermediate Value Theorem, which is not required but is correct.

In part (c) the response earned the first point with the sum of products expression given in line 1. The response would have also earned the second point for this expression in line 1, with no further simplification. In this case, correct simplification to the boxed answer of -51 in line 6 earned the second point.

In part (d) the response earned the first and second points with the correct $\frac{dV}{dt}$ expression given in line 2. The response would have also earned the third point for the correct evaluation of this expression given in line 3, with no further simplification. In this case, correct simplification presented in the boxed answer in line 6 earned the third point.

Sample: 4B
Score: 5

The response earned 5 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{-3.8 - -4.4}{10 - 7}$ at the beginning of line 1, with no simplification. In this case, correct simplification to the boxed answer of $\frac{1}{5}$ at the end of line 1 earned the first point. The response earned the second point for the correct units presented in the boxed answer in line 1.

In part (b) the response does not establish that -6 is between $r'(0)$ and $r'(3)$; thus, it did not earn the first point. The response did not earn the second point because the conclusion of "NO" is incorrect.

In part (c) the response earned the first point with the sum of products given in line 1. The response would have also earned the second point for this expression in line 1, with no further simplification. In this case, the response did not earn the second point because of an incorrect simplification to -37.0.

In part (d) the response earned the first and second points with the correct expression for $\frac{dV}{dt}$ given on line 3. The response did not earn the third point because this expression was never evaluated.

Sample: 4C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{-3.8 - (-4.4)}{10 - 7}$ in line 1, with no simplification. In this case, correct simplification to the final answer of 0.2 at the end of line 1 earned the first point. The response does not present correct units, so it did not earn the second point.

Question 4 (continued)

In part (b) the response earned the first point for the statement given in line 4: "... since -6 is between -6.1 and $-5.0.$" The response does not establish that the continuity condition of the Intermediate Value Theorem has been met and incorrectly names the theorem as the Mean Value Theorem, so it did not earn the second point.

In part (c) the response earned the first point with the sum of products given in line 1. The response would have also earned the second point for this expression in line 1, with no further simplification. In this case, incorrect simplification leads to a final answer of $-52,$ so the response did not earn the second point.

In part (d) the response earned no points. The response does not present either a correct product rule or correct chain rule; thus, it did not earn either of the first two points and is not eligible for the third point.

2022

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AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 5

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

Part B (AB): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes

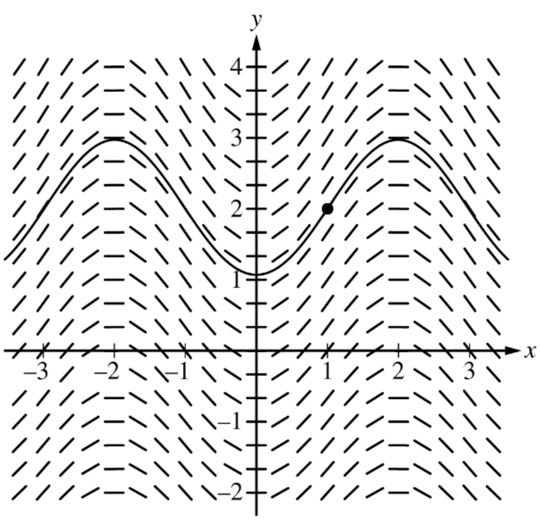
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

Model Solution	Scoring

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.



Solution curve — **1 point**

Scoring notes:

- The solution curve must pass through the point $(1, 2)$, extend reasonably close to the left and right edges of the square and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must indicate $f(x) > 0$ for all points on the curve.
- All local maximum/minimum points on the solution curve must occur at horizontal line segments in the slope field.

Total for part (a) — 1 point

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

$\left.\frac{dy}{dx}\right\|_{(x,\, y)=(1,\, 2)} = \frac{1}{2} \cdot 3 \cdot \sin\left(\frac{\pi}{2}\right) = \frac{3}{2}$ An equation for the tangent line is $y = 2 + \frac{3}{2}(x - 1)$.	Tangent line equation	**1 point**
$f(0.8) \approx 2 + \frac{3}{2}(0.8 - 1) = 1.7$	Approximation	**1 point**

Scoring notes:

- The tangent line equation can be presented in any equivalent form.
- An incorrect tangent line equation with a slope of $\frac{3}{2}$ is eligible to earn the second point for a consistent answer.
- A response of only $2 + \frac{3}{2}(0.8 - 1)$ earns the second point but not the first.

Total for part (b) 2 points

(c) It is known that $f''(x) > 0$ for $-1 \le x \le 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

Because $f''(x) > 0$, f is concave up on $-1 \le x \le 1$, the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ is an underestimate.	Answer with reason	**1 point**

Scoring notes:

- The reason must include $f''(x) > 0$, $f'(x)$ is increasing, or $f(x)$ is concave up.

Total for part (c) 1 point

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ with the initial condition $f(1) = 2$.

$\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$	Separation of variables	**1 point**
$2\sqrt{y+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right) + C$	One correct antiderivative	**1 point**
	The other correct antiderivative	**1 point**
$f(1) = 2 \Rightarrow 2\sqrt{2+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\cdot 1\right) + C$ $\Rightarrow 6 = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right) + C \Rightarrow C = 6$ $\sqrt{y+7} = 3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)$	Constant of integration and uses initial condition	**1 point**
$y = \left(3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)\right)^2 - 7$	Solves for y	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points.
- A response with no constant of integration can earn at most the first 3 points.
- A response is eligible for the fourth point only if it has earned the first point and at least 1 of the 2 antiderivative points.
    - Special case: The incorrect separation of $\sqrt{y+7}\ dy = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$ does not earn the first point, is only eligible for the antiderivative point for $-\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right)$, and is eligible for the fourth point.
- An eligible response earns the fourth point by correctly including the constant of integration in an equation and substituting 1 for x and 2 for y.
- A response is eligible for the fifth point only if it has earned the first 4 points.

Total for part (d) 5 points

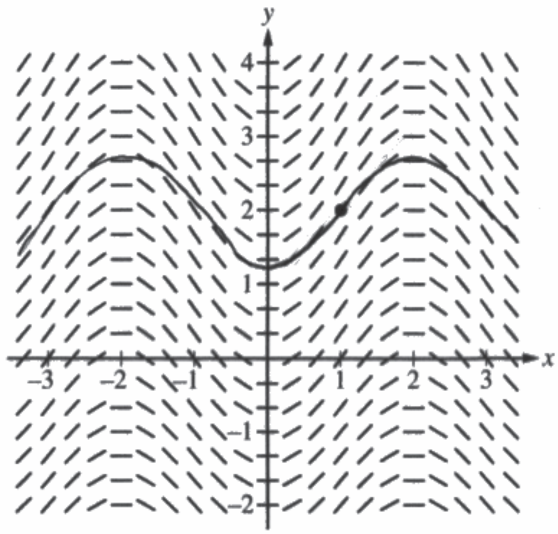
Total for question 5 9 points

Sample 5A

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)



Response for question 5(b)

$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$

At (1, 2):

$= \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\sqrt{9}$

$= \frac{1}{2}\cdot 3 = \frac{3}{2}$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{3}{2}(x-1)$

$y = \frac{3}{2}(x-1) + 2$

$y = \frac{3}{2}(.8-1) + 2$

$f(.8) = \frac{3}{2}(-.2) + 2$

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

The approximation found in part(b) is an underestimate because the graph of f' is concave upward meaning that the tangent line would go under the curve making it an underestimation of the actual value.

Response for question 5(d)

$$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$$

$$\int\frac{dy}{\sqrt{y+7}} = \int\frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$$

$$\int(y+7)^{-\frac{1}{2}}dy = \int\frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$$

$$2\sqrt{y+7} = \frac{1}{2}-\cos\left(\frac{\pi}{2}x\right)\cdot\frac{2}{\pi}+C$$

$$2\sqrt{y+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right)+C$$

$$2\sqrt{9} = \frac{-1}{\pi}\cos\left(\frac{\pi}{2}\right)+C$$

$$6 = C$$

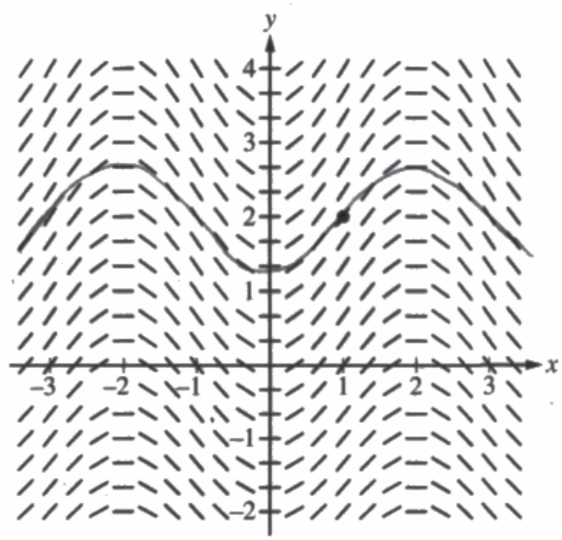
$$2\sqrt{y+7} = \frac{-1}{\pi}\cos\left(\frac{\pi}{2}x\right)+6$$

$$\boxed{y = \left(\frac{\frac{-1}{\pi}\cos\left(\frac{\pi}{2}x\right)+6}{2}\right)^2-7}$$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)



Response for question 5(b)

$y - 2 = \frac{3}{2}(x - 1)$

$y - 2 = \frac{3}{2}(0.8 - 1)$ 1.50×0.20

0.2

$= 0.25$

$= 2.20$

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

Approximation is underestimate because $f''(x) > 0$.

Response for question 5(d)

$\int \frac{1}{\sqrt{y+7}}\, dy = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\, dx$

$(y+7)^{-\frac{1}{2}}$

$\frac{(y)^{-\frac{1}{2}+1}}{2} = -\frac{1}{2}\cos\left(\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}$

$2\left(\frac{\sqrt{y}}{2}\right) = \left(-\frac{\pi}{4}\cos\left(\frac{\pi}{2}x\right)\right)\cdot 2$

$\sqrt{y}^2 = \left(-\frac{\pi}{4}\cos\left(\frac{\pi}{2}x\right)\right)^2\cdot 2^2$

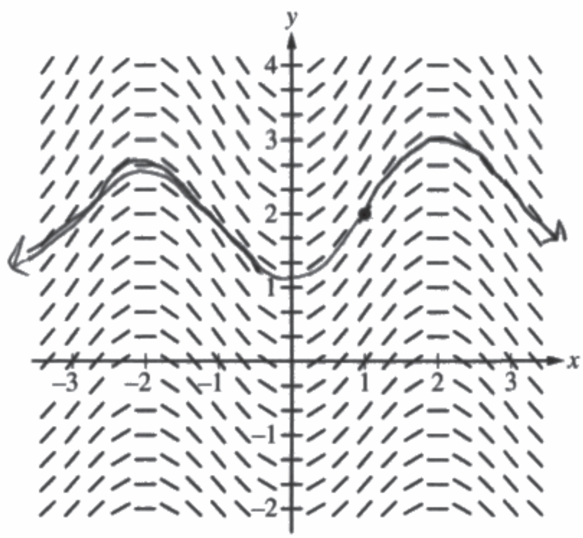
$y = \left(-\frac{\pi}{4}\cos\left(\frac{\pi}{2}x\right)\right)^2\cdot 2^2$

$y = 2$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)



Response for question 5(b)

$m = \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\sqrt{9}$

$m = \frac{1}{2}(1)(3)$

$m = \frac{3}{2}$

$y - 2 = m(x-1)$

$y - 2 = \frac{3}{2}(0.8-1)$

$y = \frac{3}{2}(-.2) + 2$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5106/12

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

Because $f''(x) > 0$ and concave up, $f(0.8)$ was an underestimate

Response for question 5(d)

$$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$$

$$\left(\frac{1}{\frac{1}{2}\sin\left(\frac{\pi}{2}x\right)} \cdot \frac{dy}{dx}\right)^2 = \sqrt{y+7}^{\,2}$$

$$-7 + \frac{1}{\frac{1}{2}\sin\left(\frac{\pi}{2}x\right)} = y$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5106/13

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem students were given a differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ and told that $y = f(x)$ is the particular solution to the equation with initial condition $f(1) = 2$. They are also told that f is defined for all real numbers.

In part (a) a portion of the slope field for this differential equation is shown, and students were asked to sketch the solution curve through the point $(1, 2)$. A correct response will draw a curve that follows the indicated slope segments in the first and second quadrants, through the point $(1, 2)$, with minimum and maximum points occurring at horizontal line segments on the slope field.

In part (b) students were asked to write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$ and to use that equation to approximate $f(0.8)$. A correct response would use the given differential equation to find the slope of the tangent line, $\left.\frac{dy}{dx}\right|_{(x,\,y)=(1,\,2)} = \frac{3}{2}$, then use this slope and the given point to find a tangent line equation of $y = 2 + \frac{3}{2}(x-1)$. Additionally, the response should substitute $x = 0.8$ in the tangent line equation to obtain an approximation of $f(0.8) \approx 1.7$.

In part (c) students were told that $f''(x) > 0$ for $-1 \le x \le 1$ and asked to reason whether the approximation found in part (b) is an over- or underestimate for $f(0.8)$. A correct response will reason that $f''(x) > 0$ on $-1 \le x \le 1$ means f is concave up on $-1 \le x \le 1$; therefore, the tangent line lies below the graph of $y = f(x)$, and the approximation is an underestimate of $f(0.8)$.

In part (d) students were asked to use separation of variables to find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 2$. A correct response should separate the variables, integrate, use the initial condition $f(1) = 2$ to determine the value of the constant of integration, and arrive at the solution of $y = \left(3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)\right)^2 - 7$.

Sample: 5A
Score: 8

The response earned 8 points: 1 point in part (a), 2 points in part (b), no points in part (c), and 5 points in part (d).

In part (a) the response earned the point with a correct solution curve passing through the point $(1, 2)$.

Question 5 (continued)

In part (b) the response would have earned the first point in line 2 on the right side for the equation $y - 2 = \frac{3}{2}(x - 1)$ but continued to simplify and earned the first point for the boxed answer in line 3 on the right side with a correct equation for the tangent line. The response would have earned the second point for the unsimplified approximation in line 4 on the right side but continued to simplify and earned the second point for the boxed answer in line 5 on the right side with a correct approximation of $f(0.8)$.

In part (c) the response correctly determines the approximation is an underestimate but did not earn the point because of the incorrect statement in line 2, "f' is concave upward."

In part (d) the response earned the first point in line 2 for a correct separation of variables. The second point was earned in line 4 on the left side of the equation for the correct antiderivative. The third point was earned in line 5 on the right side of the equation for the correct antiderivative. This antiderivative is initiated in line 4 with an unclear use of the negative sign, but this is clarified in line 5. The fourth point was earned for the correct use of "$+C$" in line 4 and for using the initial condition in line 6. The fifth point was earned for the boxed answer with a correct expression for the particular solution.

Sample: 5B
Score: 4

The response earned 4 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 1 point in part (d).

In part (a) the response earned the point with a correct solution curve passing through the point $(1, 2)$.

In part (b) the response earned the first point in line 1 with a correct equation for the tangent line. The response did not earn the second point because the approximation 2.25 is incorrect.

In part (c) the response earned the point by stating, "Approximation is underestimate because $f''(x) > 0$."

In part (d) the response earned the first point in line 1 for a correct separation of variables. The second and third points were not earned because of the incorrect antiderivatives $\frac{(y)^{-\frac{1}{2}+1}}{2}$ and $-\frac{1}{2}\cos\left(\frac{\pi}{2}x\right) \cdot \frac{\pi}{2}$ in line 3. Because neither antiderivative point was earned, the response is not eligible for the fourth or fifth points.

Sample: 5C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), no points in part (c), and no points in part (d).

In part (a) the response earned the point with a correct solution curve passing through the point $(1, 2)$. The curve does not have to be symmetric with respect to the y-axis to earn the point.

In part (b) the response earned the first point in line 1 on the left side with a correct equation for the tangent line and by clearly defining m in line 3 on the right side. The equation $y - 2 = m(x - 1)$ alone would not earn the first point. The response earned the second point for the circled approximation.

Question 5 (continued)

In part (c) the response correctly concludes "$f(0.8)$ was an underestimate" but did not earn the point because the reasoning "Because $f''(x) > 0$ and concave up" implies $f''(x)$ is concave up, which is incorrect.

In part (d) the response earned no points because there is no acceptable separation of variables.

2022

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AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 6

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

Part B (AB): Graphing calculator not allowed

Question 6 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Particle P moves along the x-axis such that, for time $t > 0,$ its position is given by $x_P(t) = 6 - 4e^{-t}$.

Particle Q moves along the y-axis such that, for time $t > 0,$ its velocity is given by $v_Q(t) = \frac{1}{t^2}$. At time $t = 1,$ the position of particle Q is $y_Q(1) = 2$.

	Model Solution	Scoring	
(a)	Find $v_P(t)$, the velocity of particle P at time t.		
	$v_P(t) = x_P'(t) = 4e^{-t}$	Answer	**1 point**

Scoring notes:

- A response that equates $x_P(t)$ with $v_P(t)$ does not earn the point.
- An unlabeled response earns the point.

Total for part (a) **1 point**

(b) Find $a_Q(t)$, the acceleration of particle Q at time t. Find all times t, for $t > 0$, when the speed of particle Q is decreasing. Justify your answer.

$a_Q(t) = v_Q{}'(t) = \frac{-2}{t^3}$	$a_Q(t)$	**1 point**
For $t > 0$, $a_Q(t) < 0$ and $v_Q(t) > 0$.	Considers signs of $a_Q(t)$ and $v_Q(t)$	**1 point**
Because the velocity and acceleration have opposite signs, the speed of particle Q is decreasing for all $t > 0$.	Answer with justification	**1 point**

Scoring notes:

- Earning the first point is not necessary for a response to be eligible to earn the second or third points; however, the response must present an expression for $a_Q(t)$ to be eligible for third point.
- A response earns the second point with either of the following statements: "$v_Q(t)$ and $a_Q(t)$ have opposite signs" or "$v_Q(t)$ and $a_Q(t)$ have the same sign." This statement, however, must be consistent with $v_Q(t)$ and the presented expression for $a_Q(t)$.
- A response must earn the second point to be eligible for the third point. The answer must be consistent with the presented justification. Furthermore, responses for which $a_Q(t) > 0$ for $t > 0$ must conclude that there is no time at which the speed of the particle is decreasing.
- A response that indicates $v_Q(t) < 0$ does not earn the third point, even if the answer and justification are consistent with a reported sign of $a_Q(t)$.

Total for part (b) 3 points

(c) Find $y_Q(t)$, the position of particle Q at time t.

$y_Q(t) = y_Q(1) + \int_1^t \frac{1}{s^2}\,ds$	Integral	**1 point**
	Uses initial condition	**1 point**
$= 2 - \left(\frac{1}{s}\Big\|_1^t\right) = 2 - \frac{1}{t} + 1 = 3 - \frac{1}{t}$	Answer	**1 point**

Scoring notes:

- A response that presents $\int_1^t \frac{1}{t^2}\,dt$ (using the same variable as a limit and integrand function) does not earn the first point unless it is followed by an attempt at integration.
- A response that presents either $\int \frac{1}{t^2}\,dt$ or $-\frac{1}{t}$ (with no integral) earns the first point. If the response continues and presents $2 = -1 + C$, then the response earns the second point.
- A response that presents only $y_Q(t) = -\frac{1}{t} + 3$ will earn all 3 points. Note that the right side of this equation suffices to earn all points. A response of $y_Q(t) = -\frac{1}{t} + C$, where $C \neq 3$, (with no additional supporting work) earns only the first point.

Total for part (c) 3 points

(d) As $t \to \infty$, which particle will eventually be farther from the origin? Give a reason for your answer.

For particle P, $\lim_{t\to\infty}\left(6 - 4e^{-t}\right) = 6$. For particle Q, $\lim_{t\to\infty}\left(3 - \frac{1}{t}\right) = 3$.	One correct limit	**1 point**
Because $6 > 3$, particle P will eventually be farther from the origin.	Answer with reason	**1 point**

Scoring notes:

- A response with an incorrect $y_Q(t)$ from part (c) is eligible for both points in part (d) provided $y_Q(t)$ is a non-constant function. The second point is earned for a consistent answer with reason, and limits correct for particle P and the presented $y_Q(t)$.
- Responses that present statements such as "$6 - 4e^{-t}$ approaches 6" or "Q goes to 3" earn the first point and are eligible for the second point.
- A response that treats ∞ as an input for $x_P(t)$ or $y_Q(t)$, such as "$6 - 4e^{-\infty}$" or "$3 - \frac{1}{\infty}$" is not eligible for the second point.

Total for part (d) 2 points

Total for question 6 9 points

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$x_p(t) = 6 - 4e^{-t}$

$V_p(t) = 4e^{-t}$

Response for question 6(b)

$V_q(t) = \frac{1}{t^2}$

$A_q(t) = \frac{-2}{t^3}$

$V_q(t)$ is positive on the interval $(0, \infty)$

$A_q(t)$ is negative on the interval $(0, \infty)$

so the speed is decreasing on the interval $(0, \infty)$

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$V_Q(t) = \frac{1}{t^2}$

$y_Q(t) = \int \frac{1}{t^2}\, dt$

$y_Q(t) = -(t)^{-1} + C$

$2 = -(1) + C$

$C = 3$

$y_Q(t) = -\frac{1}{t} + 3$

Response for question 6(d)

$\lim_{t \to \infty} y_Q(t) = 3$

$\lim_{t \to \infty} x_P(t) = 6 - 4(0)$
$= 6$

Particle P, because its end behavior is farther away from the origin than particle Q

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$x_p(t) = 6 - 4e^{-1}$

$v_p(t) = 4e^{-t}$ $\quad x'(t) = v(t)$

Response for question 6(b)

$v_q(t) = t^{-2}$ or $\frac{1}{t^2}$

$a_q(t) = -2t^{-3}$ or $\frac{-2}{t^3}$

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$\int v_a(t)\,dt$

$\int t^{-2}\,dt$

$y_a(t) = \frac{t^{-1}}{-1} + c$ or $-\frac{1}{t} + c$

$y(1) = -\frac{1}{1} + c$

$2 = -1 + c$

$+1 \quad +1$

$c = 3$

$y(t) = -\frac{1}{t} + 3$

Response for question 6(d)

$\lim_{t \to \infty} x_p(t)$

$\lim_{t \to \infty} y_a(t)$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5106/15

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$V_p(t) = \frac{d}{dt}(6 - 4e^{-t})$

$V_p(t) = 4e^{-t}$

Response for question 6(b)

$a_Q(t) = \frac{d}{dt}\left(\frac{1}{t^2}\right)$

$a_Q(t) = \frac{2t}{t^4}$

the speed of particle Q is always increasing for all $t > 0$

for all $t > 0$ $a_Q(t) =$ positive.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5105/14

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$Y_Q(t) = 2 + \int_1^t (V_Q(b))\,db$

Response for question 6(d)

$\underset{t \to \infty}{V_P(t)} = 4e^{-\infty}$

$\underset{t \to \infty}{V_Q(t)} = \frac{1}{\infty^2}$

∴ particle P will eventually be further from the origin because particle P's speed is infinatly growing to where as particle Q's Speed is infinatley decreasing

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem, for time $t > 0$, particle P is moving along the x-axis with position $x_P(t) = 6 - 4e^{-t}$. A second particle, Q, is moving along the y-axis with velocity $v_Q(t) = \frac{1}{t^2}$ and position $y_Q(1) = 2$ at time $t = 1$.

In part (a) students were asked to find the velocity of particle P at time t. A correct response would find the derivative of the given position function, $v_P(t) = 4e^{-t}$.

In part (b) students were asked to find the acceleration of particle Q at time t and then to find all times ($t > 0$) when the speed of particle Q is decreasing. A correct response should recognize that the acceleration of the particle is the derivative of the velocity, $a_Q(t) = v_Q'(t) = \frac{-2}{t^3}$, then observe that for all times $t > 0$ this acceleration is negative and the given velocity $\frac{1}{t^2}$ is positive. Therefore, the acceleration and velocity of particle Q have opposite signs and thus the speed of this particle is decreasing for all $t > 0$.

In part (c) students were asked to find the position of particle Q at time t. A correct response should integrate the given velocity function, $\int_1^t v_Q(s)\,ds = \int_1^t \frac{1}{s^2}\,ds$, and add the given initial position, $y_Q(1) = 2$, to obtain a position function of $y_Q(t) = 3 - \frac{1}{t}$.

Lastly, in part (d) students were asked to reason which particle would eventually be farther from the origin as the time t approaches infinity. A correct response should evalute the limits, as $t \to \infty$, of the position functions of both particles, $\lim_{t\to\infty} x_P(t) = 6$, and $\lim_{t\to\infty} y_Q(t) = 3$. Because $6 > 3$, particle P would eventually be farther from the origin than would be particle Q.

Sample: 6A
Score: 9

The response earned 9 points: 1 point in part (a), 3 points in part (b), 3 points in part (c), and 2 points in part (d).

In part (a) the response earned the first point in the second line with the correct velocity for particle P.

In part (b) the response earned the first point in the second line on the left with the correct acceleration of particle Q. The response earned the second point in the first two lines on the right with the correct behaviors of $v_Q(t)$ and $a_Q(t)$. The response earned the third point in the last line with the conclusion that "the speed is decreasing on the interval $(0, \infty)$."

Question 6 (continued)

In part (c) the response earned the first point in the second line on the left with the integral $\int \frac{1}{t^2}\,dt$. Note that the correct antiderivative is presented in the line below. The equation $2 = -(1) + C$ in the fourth line on the left earned the second point. The equation on the right earned the third point.

In part (d) the response earned the first point with the correct evaluation of the limit for $y_Q(t)$ in the first line on the left. The correct limit values for the positions of particles P and Q in the first line together with the conclusion in the last two lines that "Particle P, because its end behavior is farther away from the origin than particle Q," earned the second point.

Sample: 6B
Score: 5

The response earned 5 points: 1 point in part (a), 1 point in part (b), 3 points in part (c), and no points in part (d).

In part (a) the response earned the point with the correct expression for the velocity of particle P in the second line on the left.

In part (b) the response earned the first point in the second line with the correct acceleration of particle Q. Because no further work is presented, the response earned neither the second nor the third point.

In part (c) the response earned the first point with the integral expression in the first line. Note that either the equivalent expression in the second line or the correct antiderivative on the third line would have also earned the point. The response earned the second point in the fifth line with the equation $2 = -1 + C$. The response earned the third point with the correct circled answer on the right.

In part (d) the response earned neither point as neither limit has been evaluated, nor has a conclusion been drawn.

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d).

In part (a) the response earned the point in the second line with the correct expression for the velocity of particle P.

In part (b) the response did not earn the first point because the presented acceleration of particle Q is missing the negative sign. The response did not earn the second point because no consideration of the sign of $v_Q(t)$ is presented. While the statement presented that "the speed of particle Q is always increasing for all $t > 0$" is consistent with the $a_Q(t)$ presented, the response is not eligible for the third point because the second point was not earned.

In part (c) the response earned the first two points with the presented equation. Because no integration has been done, the response did not earn the third point.

In part (d) the response did not earn the first point because no limit values are presented. Because the expressions on the left treat ∞ as a function input, the response is not eligible for the second point.